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Characterizations of h-intra- and h-quasi-hemiregular hemirings

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ABSTRACT

The concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-(h-quasi-)ideals of hemirings are introduced. Some new characterization theorems of these kinds of fuzzy h-ideals are also given. In particular, some characterizations of the h-intra-hemiregular and h-quasi-hemiregular hemirings are investigated by these kinds of fuzzy h-ideals.

1. Introduction

Semirings or hemirings, regarded as a generalization of rings, have been recently found particularly useful in solving problems in different disciplines of applied mathematics and information sciences because a semiring provides an algebraic framework for modeling [1]. In applications, hemirings are useful in automata and formal languages (see [2,3]). We note that the ideals of semirings play a crucial role in the structure theory; however, according to Henriksen [4], ideals in semirings do not in general coincide with the ideals of a ring. The general properties of fuzzy k-ideals of semirings were first described in [5]. In 2004, Jun et al. [6] considered the *fuzzy h-ideals* of hemirings. The *h-hemiregular hemirings* were described by Zhan et al. by using the fuzzy h-ideals [7]. Furthermore, Yin et al. introduced the concepts of fuzzy h-bi-ideals and fuzzy h-quasi-ideals of hemirings in [8]. As a continuation of these investigations, Ma and Zhan [9] introduced the concepts of $(\in, \in \vee q)$-fuzzy h-bi-ideals (resp., h-quasi-ideals) of a hemiring and investigated some of their properties. The other important results related with fuzzy k-ideals and h-ideals of a hemiring were given in [10–23].

After the concept of fuzzy sets introduced by Zadeh [24,25], there have been a number of generalizations of this fundamental concept. A new type of fuzzy subgroup, that is, the $(\in, \in \vee q)$-fuzzy subgroup, was introduced by Bhakat and Das in [26] by using the combined notions of "belongingness" and "quasi-coincidence" of fuzzy points and fuzzy sets. It is natural to investigate some similar generalizations of the existing fuzzy subsystems by considering some other structures. With this objective in mind, Davvaz et al. in [27] obtained some results in near-rings. In particular, Ma et al. discussed the properties of generalized interval-valued fuzzy h-ideals of hemirings in [28]. Fruitful results have been obtained in the literature.

This paper is organized as follows. In Section 2, we first give some basic definitions and results of hemirings. Then in Sections 3 and 4, we introduce the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-(h-quasi-)ideals of hemirings, respectively. Some new characterization theorems of fuzzy h-ideals of a hemiring are given. In Section 5, we investigate some characterizations of the h-intra-hemiregular hemirings. Finally, we discuss some characterizations of the h-quasi-hemiregular hemirings in Section 6.

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2. Preliminaries

Recall that a *semiring* is an algebraic system $(S, +, \cdot)$ consisting of a non-empty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that $(S, +)$ and $(S, \cdot)$ are semigroups and the following distributive laws

$$a(b + c) = ab + ac \quad \text{and} \quad (a + b)c = ac + bc$$

are satisfied for all $a, b, c \in S$.

By *zero* of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and $0 + x = x + 0 = x$ for all $x \in S$. A semiring with zero and a commutative semigroup $(S, +)$ is called a *hemiring*.

Throughout this paper, S is a hemiring. We also write $a \wedge b$ for $\min\{a, b\}$ and $a \vee b$ for $\max\{a, b\}$, where a and b are real numbers.

A *left (resp., right) ideal* of a semiring is a subset A of S closed with respect to the addition and such that $SA \subseteq A$ (resp., $AS \subseteq A$). A subset A is called an *ideal* if it is both a left ideal and a right ideal of S. A subset B of a semiring S is called a *bi-ideal* of S if B is closed under addition and multiplication such that $BSB \subseteq B$. A subset Q of a semiring S is called a *quasi-ideal* of S if Q is closed under addition and $SQ \cap QS \subseteq Q$.

A left ideal (right ideal, ideal and bi-ideal) A of S is called a *left h-ideal (right h-ideal, h-ideal and h-bi-ideal)*, respectively, if for any $x, z \in S$ and $a, b \in A$, $x + a + z = b + z \to x \in A$.

The *h-closure* $\overline{A}$ of A in S is defined by

$$\overline{A} = \{x \in S \mid x + a_1 + z = a_2 + z \text{ for some } a_1, a_2 \in A, z \in S\}.$$

Clearly, if A is a left ideal of S, then $\overline{A}$ is the smallest left h-ideal of S containing A. We also have $\overline{\overline{A}} = \overline{A}$ for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{A} \subseteq \overline{B}$. A quasi-ideal Q of S is called an h-quasi-ideal of S if $\overline{SQ} \cap \overline{QS} \subseteq Q$ and for any $x, z \in S$ and $a, b \in Q$ from $x + a + z = b + z$, it follows $x \in Q$.

We next state some fuzzy logic concepts. Recall that a fuzzy set is a function $\mu : S \to [0, 1]$. We denote by $\mathcal{F}(S)$ the set of all fuzzy sets of S. For any $A \subseteq S$, we denote the characteristic function of A by χ_A.

A fuzzy set μ of S of the form

$$\mu(y) = \begin{cases} t\ (\neq 0) & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a *fuzzy point with support x and value t* and is denoted by x_t.

A fuzzy point x_t is said to be "*belong to*" (resp., "*quasi-coincident with*") a fuzzy set μ, written as $x_t \in \mu$ (resp., $x_t q \mu$) if $\mu(x) \geq t$ (resp., $\mu(x) + t > 1$).

If $x_t \in \mu$ or $x_t\, q\, \mu$, then we write $x_t \in \vee q\mu$. If $\mu(x) < t$ (resp., $\mu(x) + t \leq 1$), then we say that $x_t \overline{\in} \mu$ (resp., $x_t\, \overline{q}\, \mu$).

We note here that the symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For a fuzzy point x_r and μ of S, we say

(1) $x_r \in_\gamma \mu$ if $\mu(x) \geq r > \gamma$.
(2) $x_r q_\delta \mu$ if $\mu(x) + r > 2\delta$.
(3) $x_r \in_\gamma \vee q_\delta \mu$ if $x_r \in_\gamma \mu$ or $x_r q_\delta \mu$.

For any two fuzzy sets μ, ν of S and $\gamma, \delta \in [0, 1]$, $\gamma < \delta$. Define a new ordered relation "$\subseteq \vee q_{(\gamma,\delta)}$" as follows:

$$x_r \in_\gamma \mu \Longrightarrow x_r \in_\gamma \vee q_\delta \nu \quad \text{for all } x \in S \text{ and } r \in (\gamma, r].$$

Define a relation "$\sim$" on $\mathcal{F}(S)$ by

$$\mu \sim \nu \Longleftrightarrow \mu \subseteq \vee q_{(\gamma,\delta)} \nu \text{ and } \nu \subseteq \vee q_{(\gamma,\delta)} \mu.$$

Lemma 2.1. *Let μ and ν be any two fuzzy sets of S. Then $\mu \subseteq \vee q_{(\gamma,\delta)} \nu \Longleftrightarrow \nu(x) \vee \gamma \geq \mu(x) \wedge \delta$ for all $x \in S$.*

Proof. Let $\mu \subseteq \vee q_{(\gamma,\delta)} \nu$. If there exists $x \in S$ such that $\nu(x) \vee \gamma < r < \mu(x) \wedge \delta$, that is, $x_r \in_\gamma \mu$, but $x_r \overline{\in_\gamma \vee q_\delta} \nu$, a contradiction.

Conversely, let $\nu(x) \vee \gamma \geq \mu(x) \wedge \delta$ for all $x \in S$. If $\mu \overline{\subseteq \vee q_{(\gamma,\delta)}} \nu$, then there exists $x \in S$ and $r > \gamma$ such that $x_r \in_\gamma \mu$, but $x_r \overline{\in_\gamma \vee q_\delta} \nu$, and so $\mu(x) \geq r$, $\nu(x) < r$ and $\nu(x) + r < 2\delta$. Thus, $\nu(x) \vee \gamma < \mu(x) \wedge \delta$, a contradiction. □

The following is obvious.

Lemma 2.2. *$\mu \subseteq \vee q_{(\gamma,\delta)} \nu \subseteq \vee q_{(\gamma,\delta)} \omega \Longrightarrow \mu \subseteq \vee q_{(\gamma,\delta)} \omega$.*

Note that Lemma 2.1 gives that

$$\mu \sim \nu \Longleftrightarrow (\mu(x) \wedge \delta) \vee \gamma = (\nu(x) \wedge \delta) \vee \gamma$$

for all $x \in S$ and it follows from Lemma 2.1 to 2.2 that "$\sim$" is an equivalence.

Definition 2.3 ([8,23]). Let μ and ν be fuzzy sets of S.

(i) The h-sum of μ and ν is

$$(\mu +_h \nu)(x) = \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \mu(a_1) \wedge \mu(a_2) \wedge \nu(b_1) \wedge \nu(b_2)$$

and $(\mu +_h \nu)(x) = 0$ if x cannot be expressed as $x + a_1 + b_1 + z = a_2 + b_2 + z$.

(ii) The h-product of μ and ν is

$$(\mu \odot_h \nu)(x) = \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z} \mu(a_i) \wedge \mu(a'_j) \wedge \nu(b_i) \wedge \nu(b'_j)$$

and $(\mu \odot_h \nu)(x) = 0$ if x cannot be expressed as $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$.

Proposition 2.4 ([8,23]). *Let $A, B \subseteq S$. Then we have*

(1) $A \subseteq B \Leftrightarrow \chi_A \subseteq \vee q_{(\gamma,\delta)}\ \chi_B$;
(2) $\chi_A \cap \chi_B = \chi_{A\cap B}$;
(3) $\chi_A \odot_h \chi_B = \chi_{\overline{AB}}$;
(4) $\chi_A +_h \chi_B = \chi_{\overline{A+B}}$.

3. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideals

In this section, we introduce the concept of fuzzy h-bi-ideals in hemirings. Some new characterization theorems of fuzzy h-bi-ideals of a hemiring are given.

Definition 3.1. Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. A fuzzy set μ of S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal of S if for all $t, r \in (\gamma, 1]$ and $x, z \in S$,

(F1a) $\mu +_h \mu \subseteq \vee q_{(\gamma,\delta)}\mu$;
(F1b) $\chi_S \odot_h \mu \subseteq \vee q_{(\gamma,\delta)}\mu$ (resp., $\mu \odot_h \chi_S \subseteq \vee q_{(\gamma,\delta)}\mu$);
(F1c) $x + a + z = b + z,\ a_t, b_r \in_\gamma \mu \Longrightarrow x_{t\wedge r} \in_\gamma \vee q_\delta \mu$ for all $a, b, x, z \in S$.

A fuzzy set of S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal if it is both an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal.

Example 3.2. Let $S = \{0, a, b\}$ be a hemiring with the Cayley table as follows:

+	0	a	b
0	0	a	b
a	a	0	b
b	b	b	0

.	0	a	b
0	0	0	0
a	0	0	0
b	0	0	b

(see [23]). Define a fuzzy set μ of S by $\mu(0) = 0.6$, $\mu(a) = 0.8$ and $\mu(b) = 0.2$. Then μ is an $(\in_{0.6}, \in_{0.6} \vee q_{0.8})$-fuzzy h-ideal of S, but it is not a fuzzy h-ideal of S.

Definition 3.3. Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. A fuzzy set μ of S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S if it satisfies (F1a), (F1c) and

(F2a) $\mu \odot_h \mu \subseteq \vee q_{(\gamma,\delta)}\mu$;
(F2b) $\mu \odot_h \chi_S \odot_h \mu \subseteq \vee q_{(\gamma,\delta)}\mu$.

Example 3.4. Let $\mathbb{N}$ and $\mathbb{P}$ be sets of all positive integers and positive real numbers, respectively. Consider

$$S = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} | a, b \in \mathbb{P}, c \in \mathbb{N} \right\} \bigcup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

then S is a hemiring.

Define R and L be the sets of $\left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} | a, b \in \mathbb{P}, c \in \mathbb{N}, a < b \right\} \bigcup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} p & 0 \\ q & k \end{pmatrix} | p, q \in \mathbb{P}, k \in \mathbb{N}, q > 3 \right\} \bigcup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$, respectively. One can check that R and L are a right and a left h-ideal of S, respectively. Then the product RL is an h-bi-ideal of S (see [8]). By routine calculation, the characteristic function χ_{RL} of RL is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S.

Theorem 3.5. *A fuzzy set μ of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S if and only if it satisfies:*
(F3a) $\mu(x+y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ *for all* $x, y \in S$;
(F3b) $\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta$ *for all* $x, y \in S$;
(F3c) $\mu(xyz) \vee \gamma \geq \mu(x) \wedge \mu(z) \wedge \delta$ *for all* $x, y, z \in S$;
(F3d) $x + a + z = b + z \Longrightarrow \mu(x) \vee \gamma \geq \mu(a) \wedge \mu(b) \wedge \delta$.

Proof. Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S. If there exist $x, y \in S$ such that $\mu(x+y) \vee \gamma < t < \mu(x) \wedge \mu(y) \wedge \delta$, then $\mu(x) > t, \mu(y) > t, \mu(x+y) < t < \delta$, that is, $(x+y)_t \overline{\in_\gamma \vee q_\delta} \mu$. On the other hand, $\mu(0) \vee \gamma \geq \mu(x) \wedge \delta \geq t \wedge \delta = t > \gamma$, and so $\mu(0) \geq \mu(x) \wedge \delta$. Thus,

$$
\begin{aligned}
(\mu +_h \mu)(x+y) &= \bigvee_{x+y+a_1+b_1+z=a_2+b_2+z} \mu(a_1) \wedge \mu(a_2) \wedge \mu(b_1) \wedge \mu(b_2) \\
&\geq \mu(0) \wedge \mu(x) \wedge \mu(y) \\
&\geq \mu(x) \wedge \mu(y) \wedge \delta \\
&> t \wedge \delta = t,
\end{aligned}
$$

which implies, $(x+y)_t \in_\gamma \mu +_h \mu$, and hence $(x+y)_t \in_\gamma \vee q_\delta \mu$, contradiction. Thus (F3a) holds.

Similarly, we can (F3b) holds.

If there exist $x, y, z \in S$ such that $\mu(xyz) \vee \gamma < t < \mu(x) \wedge \mu(z) \wedge \delta$, then $\mu(x) > t, \mu(z) > t$,and $\mu(xyz) < t < \delta$, that is, $(xyz)_t \overline{\in_\gamma \vee q_\delta} \mu$. On the other hand, we can check $\mu \odot_h \chi_S$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal of S and so $(\mu \odot \chi_S)(0) \vee \gamma \geq (\mu \odot_h \chi_S)(xy) \wedge \delta$. Thus

$$
\begin{aligned}
(\mu \odot_h \chi_S \odot_h \mu)(xyz) \vee \gamma &= \bigvee_{xyz+\sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a'_j b'_j + z'} (\mu \odot_h \chi_S)(a_i) \wedge (\mu \odot_h \chi_S)(a'_j) \wedge (\mu(b_i) \wedge (\mu(b'_j)) \vee \gamma \\
&\geq ((\mu \odot \chi_S)(0) \wedge (\mu \odot_h \chi_S)(xy) \wedge \mu(0) \wedge \mu(z)) \vee \gamma \\
&= ((\mu \odot \chi_S)(0) \vee \gamma) \wedge ((\mu \odot_h \chi_S)(xy) \vee \gamma) \wedge (\mu(0) \vee \gamma) \wedge (\mu(z) \vee \gamma) \\
&\geq (\mu \odot \chi_S)(xy) \wedge \delta) \wedge (\mu(z) \wedge \delta) \\
&= (\mu \odot \chi_S)(xy) \wedge \mu(z) \wedge \delta \\
&= ((\mu \odot \chi_S)(xy) \vee \gamma) \wedge \mu(z) \wedge \delta \\
&= \bigvee_{xy+\sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a'_j b'_j + z'} (((\mu(a_i) \wedge \mu(a'_j)) \vee \gamma) \wedge (\mu(z) \wedge \delta) \vee \gamma) \wedge \mu(z) \wedge \delta \\
&\geq ((\mu(x) \wedge \mu(0)) \vee \gamma) \wedge (\mu(z) \wedge \delta) \\
&\geq (\mu(x) \wedge \delta) \wedge (\mu(z) \wedge \delta) \\
&= \mu(x) \wedge \mu(z) \wedge \delta \\
&\geq t,
\end{aligned}
$$

which implies, $(xyz)_t \in_\gamma \mu \odot_h \chi_S \odot_h \mu$, contradiction. Thus (F3c) holds.

Finally, if there exist $a, b, x, z \in S$ with $x + a + z = b = z$ and $\gamma \in [0, 1]$ such that $\mu(x) \vee \gamma < r < \mu(a) \wedge \mu(b) \wedge \delta$, then $\mu(a) \geq r, \mu(b) \geq r$ and $\mu(x) < r < \delta$, and so $a_r, b_r \in_\gamma \mu$ and $x_r \overline{\in_\gamma \vee q_\delta} \mu$, contradiction. This proves (F3d) holds.

Conversely, assume that the conditions (F3a)–(F3d) hold. If $x_t \in_\gamma \mu +_h \mu$, but $x_t \overline{\in_\gamma \vee q_\delta} \mu$, then $\mu(x) < t$ and $\mu(x) < \delta$. For any $a_1, a_2, b_1, b_2, x, z \in S$ such that $x + a_1 + b_1 + z = a_2 + b_2 + z$, then by (F3a) and (F3c), we have

$$
\begin{aligned}
\delta > \mu(x) \vee \gamma &\geq \mu(a_1 + b_1) \wedge \mu(a_2 + b_2) \wedge \delta \\
&\geq (\mu(a_1 + b_1) \vee \gamma) \wedge (\mu(a_2 + b_2) \vee \gamma) \wedge \delta \\
&\geq \mu(a_1) \wedge \mu(b_1) \wedge \mu(a_2) \wedge \mu(b_2) \wedge \delta,
\end{aligned}
$$

which implies, $\mu(x) \geq \mu(a_1) \wedge \mu(b_1) \wedge \mu(a_2) \wedge \mu(b_2)$. Thus,

$$
\begin{aligned}
t \leq (\mu +_h \mu)(x) &= \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \mu(a_1) \wedge \mu(a_2) \wedge \mu(b_1) \wedge \mu(b_2) \\
&= \bigvee_{x+a_1+b_1+z=a_2+b_2+z} \mu(x) \\
&= \mu(x),
\end{aligned}
$$

contradiction. Thus (F1a) holds.

Similarly, we can prove (F2a) and (F2b) hold.

Finally, if there exist $a, b, x, z \in S$ and $t, r \in [0, 1]$ with $x + a + z = b + z$ and $a_t, b_r \in_\gamma \mu$ such that $x_{t \wedge r} \overline{\in_\gamma \vee q_\delta} \mu$, then $\mu(a) \geq t, \mu(b) \geq r, \mu(x) \vee \gamma < t \wedge r < \delta$, contradiction. Thus, (F1c) holds. $\square$

Remark 3.6. For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal μ of S, we can

(i) If $\gamma = 0$ and $\delta = 1$, then μ is the fuzzy h-bi-ideal of S (see [8]).
(ii) If $\gamma = 0$ and $\delta = 0.5$, then μ is the $(\in, \in \vee q)$-fuzzy h-bi-ideal of S (see [9]).

For any fuzzy set μ of S, we define $\mu_r^\gamma = \{x \in S \mid x_r \in_\gamma \mu\}$, $\mu_r^\delta = \{x \in S \mid x_r q_\delta \mu\}$ and $[\mu]_r^\delta = \{x \in S \mid x_r \in_\gamma \vee q_\delta \mu\}$ for all $r \in [0, 1]$. It is clear that $[\mu]_r^\delta = \mu_r^\gamma \cup \mu_r^\delta$.

The next theorem provides the relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideals of S and the crisp h-bi-ideals of S.

Theorem 3.7. *Let μ be a fuzzy set of S. Then*

(1) μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S if and only if μ_r^γ $(\neq\emptyset)$ is an h-bi-ideal of S for all $r \in (\gamma, \delta]$.

(2) If $2\delta = 1+\gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S if and only if μ_r^δ $(\neq\emptyset)$ is an h-bi-ideal of S for all $r \in (\delta, 1]$.

(3) If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S if and only if $[\mu]_r^\delta$ $(\neq\emptyset)$ is an h-bi-ideal of S for all $r \in (\gamma, 1]$.

Proof. (1) Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S.

(i) If $x, y \in \mu_r^\gamma$ and $z \in S$ for all $r \in (\gamma, \delta]$, then $\mu(x) \geq r$ and $\mu(y) \geq r$. It follows that

$$\mu(x+y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \geq r \wedge \delta > r > \gamma,$$
$$\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \geq r \wedge \delta > r > \gamma,$$
$$\mu(xzy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \geq r \wedge \delta > r > \gamma,$$

which imply $(x+y)_r, (x+y)_r, (xzy)_r \in_\gamma \mu$.

Let $x, z \in S$ and $a, b \in \mu_r^\gamma$ be such that $x + a + z = b + z$. Then $\mu(x) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta \geq r \wedge \delta = r > \gamma$, and so $\mu(x) \geq r$, that is $x_r \in_\gamma \mu$. Thus, μ_r^γ is an h-bi-ideal of S.

Conversely, assume that μ_r^γ is an h-bi-ideal of S for all $r \in (\gamma, \delta]$. Let $x, y \in S$. If $\mu(x+y) \vee \gamma < r = \mu(x) \wedge \mu(y) \wedge \delta$, then $x_r \in_\gamma \mu$,$y_r \in_\gamma \mu$, but$(x+y)_r \overline{\in_\gamma \vee q_\delta}\mu$, and so $x, y \in \mu_r^\gamma$. Since μ_r^γ is an h-bi-ideal of S, we have $x + y \in \mu_r^\gamma$, a contradiction. Thus (F3a) holds. Similarly, we can prove (F3b)–(F3d) hold.

(2) It is similar to (1).

(3) Let μ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S and $r \in (\gamma, 1]$. Then for all $x, y \in [\mu]_r^\delta$, we have $x_r, y_r \in_\gamma \vee q_\delta \mu$, that is, $\mu(x) \geq r > \gamma$ or $\mu(x) > 2\delta - r > 2\delta - 1 = \gamma$, and $\mu(y) \geq r > \gamma$ or $\mu(y) > 2\delta - r > 2\delta - 1 = \gamma$. Since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S, then $\mu(x+y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta > \gamma \wedge \delta = \gamma$. And so $\mu(x+y) \geq \mu(x) \wedge \mu(y) \wedge \delta$.

Case 1: $r \in (\gamma, \delta]$. Then $2\delta - r \geq \delta \geq r$, and so
$\mu(x+y) \geq r \wedge r \wedge \delta = r$
or $\mu(x+y) \geq r \wedge (2\delta - r) \wedge \delta = r$
or $\mu(x+y) \geq (2\delta - r) \wedge (2\delta - r) \wedge \delta = \delta > r$. Hence, $(x+y)_r \in_\gamma \mu$.

Case 2: $r \in (\delta, 1]$. Then $2\delta - r < \delta < r$, and so
$\mu(x+y) \geq r \wedge r \wedge \delta = \delta > 2\delta - r$
or $\mu(x+y) \geq r \wedge (2\delta - r) \wedge \delta = 2\delta - r$
or $\mu(x+y) \geq (2\delta - r) \wedge (2\delta - r) \wedge \delta = 2\delta - r$. Hence, $(x+y)_r q_\delta \mu$.

Thus in any case, $(x+y)_r \in_\gamma \vee q_\delta \mu$, that is, $(x+y)_r \in [\mu]_r^\delta$. Similarly, we can prove the other conditions of h-bi-ideals. Hence, $[\mu]_r^\delta$ is an h-bi-ideal of S.

Conversely, assume that $[\mu]_r^\delta$ is an h-bi-ideal of S for all $r \in (\gamma, \delta]$. If $x, y \in S$ such that $\mu(x+y) \vee \gamma < r = \mu(x) \wedge \mu(y) \wedge \delta$, then $x_r \in_\gamma \mu$, $y_r \in_\gamma \mu$, but $(x+y)_r \overline{\in_\gamma \vee q_\delta}\mu$, and so $x, y \in [\mu]_r^\delta$. Since $[\mu]_r^\delta$ is an h-bi-ideal of S, we have $x + y \in [\mu]_r^\delta$, a contradiction. Thus (F3a) holds.

Similarly, we can prove (F3b)–(F3d) hold. Hence μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S. $\square$

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 3.7, we can conclude the following results:

Corollary 3.8. *Let μ be a fuzzy set of S. Then*

(1) μ is an $(\in, \in \vee q)$-fuzzy h-bi-ideal of S if and only if μ_r $(\neq\emptyset)$ is an h-bi-ideal of S for all $r \in (0, 0.5]$ (see [9]).

(2) μ is an $(\in, \in \vee q)$-fuzzy h-bi-ideal of S if and only if $Q(\mu; r)$ $(\neq\emptyset)$ is an h-bi-ideal of S for all $r \in (0.5, 1]$, where $Q(\mu, r) = \{x \in S \mid x_r q \mu\}$.

(3) μ is an $(\in, \in \vee q)$-fuzzy h-bi-ideal of S if and only if $[\mu]_r$ $(\neq\emptyset)$ is an h-bi-ideal of S for all $r \in (0, 1]$ (see [9]).

4. $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy *h*-quasi-ideals

In this section, we introduce the concept of fuzzy h-quasi-ideals in hemirings. Some new characterization theorems of fuzzy h-quasi-ideals of a hemiring are given.

Definition 4.1. Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. A fuzzy set μ of S is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S if it satisfies (F1a), (F1c) and

(F4a) $\chi_S \odot_h \mu \cap \mu \odot_h \chi_S \subseteq \vee q_{(\gamma,\delta)}\mu$.

Example 4.2. Consider the hemiring $(N_0, +, \cdot)$, where N_0 is the set of all non-negative integers. Define a fuzzy set μ of N_0 by

$$\mu(x) = \begin{cases} 0.6 & \text{if } x \in \langle 6 \rangle, \\ 0.8 & \text{if } x \in \langle 3 \rangle - \langle 6 \rangle, \\ 0.2 & \text{otherwise} \end{cases}$$

(see [9]). Then, one easily check μ is an $(\in_{0.6}, \in_{0.6} \vee q_{0.8})$-fuzzy h-quasi-ideal of N_0.

Theorem 4.3. *A fuzzy set μ of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S if and only if it satisfies* (F3a), (F3c) *and*
(F5a) $\mu(x) \vee \gamma \geq (\chi_S \odot_h \mu)(x) \wedge (\mu \odot_h \chi_S)(x) \wedge \delta$ *for all $x \in S$;*

Proof. It is similar to Theorem 3.5. $\square$

Remark 4.4. For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal μ of S, we can
(i) If $\gamma = 0$ and $\delta = 1$, then μ is the fuzzy h-quasi-ideal of S (see [8]).
(ii) If $\gamma = 0$ and $\delta = 0.5$, then μ is the $(\in, \in \vee q)$-fuzzy h-quasi-ideal of S (see [9]).

The next theorem provides the relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideals of S and the crisp h-quasi-ideals of S.

Theorem 4.5. *Let μ be a fuzzy set of S. Then*
(1) *μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S if and only if μ_r^γ $(\neq\emptyset)$ is an h-quasi-ideal of S for all $r \in (\gamma, \delta]$.*
(2) *If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S if and only if μ_r^δ $(\neq\emptyset)$ is an h-quasi-ideal of S for all $r \in (\delta, 1]$.*
(3) *If $2\delta = 1 + \gamma$, then μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S if and only if $[\mu]_r^\delta$ $(\neq\emptyset)$ is an h-quasi-ideal of S for all $r \in (\gamma, 1]$.*

Proof. It is similar to Theorem 3.7. $\square$

If we take $\gamma = 0$ and $\delta = 0.5$ in Theorem 4.5, we can conclude the following results:

Corollary 4.6. *Let μ be a fuzzy set of S. Then*
(1) *μ is an $(\in, \in \vee q)$-fuzzy h-quasi-ideal of S if and only if μ_r $(\neq\emptyset)$ is an h-quasi-ideal of S for all $r \in (0, 0.5]$ (see [9]).*
(2) *μ is an $(\in, \in \vee q)$-fuzzy h-quasi-ideal of S if and only if $Q(\mu; r)$ $(\neq\emptyset)$ is an h-quasi-ideal of S for all $r \in (0.5, 1]$, where $Q(\mu, r) = \{x \in S \mid x_r q \mu\}$.*
(3) *μ is an $(\in, \in \vee q)$-fuzzy h-quasi-ideal of S if and only if $[\mu]_r$ $(\neq\emptyset)$ is an h-quasi-ideal of S for all $r \in (0, 1]$ (see [9]).*

The following proposition is obvious and we omit the details.

Proposition 4.7. (i) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal.*
(ii) *Every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal.*

5. *h*-intra-hemiregular hemirings

In this section, we show that the h-intra-hemiregular hemirings can be described by using these kinds of fuzzy h-ideals.

Definition 5.1 (*[7]*)**.** A hemiring S is said to be h-hemiregular if for each $x \in S$, there exist $a, a', z \in S$ such that $x + xax + z = xa'x + z$.

Definition 5.2 (*[8]*)**.** A hemiring S is said to be h-intra-hemiregular if for each $x \in S$, there exist $c_i, c_j, d_i, d_j', z \in S$ such that $x + \sum_{i=1}^m c_i x^2 d_i + z = \sum_{j=1}^n c_j' x^2 d_j' + z$. Equivalent definitions: (1) $x \in \overline{Sx^2S}$, $\forall x \in S$, (2) $A \subseteq \overline{SA^2S}$, $\forall A \subseteq S$.

Lemma 5.3 (*[8]*)**.** *Let S be a hemiring. Then the following are equivalent:*
(1) *S is h-intra-hemiregular;*
(2) *$L \cap R \subseteq \overline{LR}$ for every left h-ideal L and every right h-ideal R of S.*

Theorem 5.4. *Let S be a hemiring. Then the following are equivalent:*
(1) *S is h-intra-hemiregular;*
(2) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$, for any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal μ and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal ν.*

Proof. (1) $\Longrightarrow$ (2): Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal of S, respectively. For $x \in S$. Since S is h-intra-hemiregular, then there exist $c_i, c'_j, d_i, d'_j, z \in S$ such that

$$x + \sum_{i=1}^{m} c_i x^2 d_i + z = \sum_{j=1}^{n} c'_j x^2 d'_j + z,$$

that is, $x + \sum_{i=1}^{m}(c_i x)(x d_i) + z = \sum_{j=1}^{n}(c'_j x)(x d'_j) + z$.

Thus, we have

$$\begin{aligned}(\mu \odot_h \nu)(x) \vee \gamma &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a'_j b'_j + z'} (\mu(a_i) \wedge \mu(a'_j) \wedge \nu(b_i) \wedge \nu(b'_j)) \vee \gamma \\ &\geq (\mu(c_i x) \wedge \mu(c'_j x) \wedge \nu(x d_i) \wedge \nu(x d'_j)) \vee \gamma \\ &= (\mu(c_i x) \vee \gamma) \wedge (\mu(c'_j x) \vee \gamma) \wedge (\nu(x d_i) \vee \gamma) \wedge (\nu(x d'_j) \vee \gamma) \\ &\geq (\mu(x) \wedge \delta) \wedge (\nu(x) \wedge \delta) \\ &= \mu(x) \wedge \nu(x) \wedge \delta \\ &= (\mu \cap \nu)(x) \wedge \delta,\end{aligned}$$

which implies, $\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$.

(2) $\Longrightarrow$ (1): Let L and R be any left h-ideal and any right h-ideal of S, respectively. Then, χ_L and χ_R are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal of S, respectively.

Thus,

$$\chi_{L\cap R} = \chi_L \cap \chi_R \subseteq \vee q_{(\gamma,\delta)} \chi_L \odot_h \chi_R = \chi_{\overline{LR}},$$

which implies, $L \cap R \subseteq \overline{LR}$. It follows from Lemma 5.3 that S is h-intra-hemiregular. $\square$

Lemma 5.5 ([8]). *Let S be a hemiring. Then the following are equivalent:*

(1) *S is both h-hemiregular and h-intra-hemiregular;*
(2) *$B = \overline{B^2}$ for every h-bi-ideal B of S;*
(3) *$Q = \overline{Q^2}$ for every h-quasi-ideal Q of S.*

Theorem 5.6. *Let S be a hemiring. Then the following are equivalent:*

(1) *S is both h-hemiregular and h-intra-hemiregular;*
(2) *$\mu \sim \mu \odot_h \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal μ of S;*
(3) *$\mu \sim \mu \odot_h \mu$ for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal μ of S.*

Proof. (1) $\Longrightarrow$ (2): Let μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S. For $x \in S$. Since S is both h-hemiregular and h-intra-hemiregular, then there exist $a_1, a_2, p_i, p'_i, q_j, q'_j, z \in S$ such that

$$\begin{aligned}&x + \sum_{j=1}^{n}(x a_2 q_j x)(x q'_j a_1 x) + \sum_{j=1}^{n}(x a_1 q_j x)(x q'_j a_2 x) + \sum_{i=1}^{m}(x a_1 p_i x)(x p'_i a_1 x) + \sum_{i=1}^{m}(x a_2 p_i x)(x p'_i a_2 x) + z \\ &\quad = \sum_{i=1}^{m}(x a_2 p_i x)(x p'_i a_1 x) + \sum_{i=1}^{m}(x a_1 p_i x)(x p'_i a_2 x) + \sum_{j=1}^{n}(x a_1 q_j x)(x p'_i a_1 x) + \sum_{j=1}^{n}(x a_2 q_j x)(x q_j i' a_2 x) + z.\end{aligned}$$

Thus, we have

$$\begin{aligned}(\mu \odot_h \mu)(x) \vee \gamma &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a'_j b'_j + z'} (\mu(a_i) \wedge \mu(a'_j) \wedge \mu(b_i) \wedge \mu(b'_j)) \vee \gamma \\ &\geq (\mu(x a_2 q_j x) \wedge \mu(x q'_j a_1 x) \wedge \mu(x a_1 q_j x) \wedge \mu(x q'_j a_2 x) \\ &\quad \wedge \mu(x a_1 p_i x) \wedge \mu(x p'_i a_1 x) \wedge \mu(x a_2 p_i x) \wedge \mu(x p'_i a_2 x)) \vee \gamma \\ &= \mu(x) \wedge \delta,\end{aligned}$$

which implies, $\mu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \mu$. On the other hand, $\mu \odot_h \mu \subseteq \vee q_{(\gamma,\delta)} \mu$, whence, $\mu \sim \mu \odot_h \nu$.

(2) $\Longrightarrow$ (3): It is clear from Proposition 4.7.

(3) $\Longrightarrow$ (1): Let Q be any h-quasi-ideal of S. Then, χ_Q is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S.

Thus,

$$\chi_Q \sim \chi_Q \odot_h \chi_Q = \chi_{\overline{Q^2}},$$

which implies, $Q = \overline{Q^2}$. It follows from Lemma 5.5 that S is both h-hemiregular and h-intra-hemiregular. $\square$

Similarly, we can conclude the following result.

Theorem 5.7. *Let S be a hemiring. Then the following are equivalent:*
(1) *S is both h-hemiregular and h-intra-hemiregular;*
(2) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideals μ and ν of S;*
(3) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal ν of S;*
(4) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal ν of S;*
(5) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$, for all $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideals μ and ν of S.*

6. *h*-quasi-hemiregular hemirings

In this section, we show that the h-quasi-hemiregular hemirings can be described by using these kinds of fuzzy h-ideals.

Definition 6.1. A subset A of S is called idempotent if $A = \overline{A}_2$. A fuzzy set μ is called $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy idempotent if $\mu \sim \mu \odot_h \mu$.

Definition 6.2. A hemiring S is called left (resp., right) h-quasi-hemiregular if every left (resp., right) h-ideal is idempotent, and is called h-quasi-hemiregular if every left h-ideal and every right h-ideal is idempotent.

Example 6.3. Let $S = \{0, a, b, c\}$ be a hemiring with the Cayley table as follows:

$+$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	c	a
c	c	c	a	b

$\cdot$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	a	a
c	0	a	a	a

One can easily check that $\{0, a\}$ and $\{0, a, b, c\}$ are all left (resp., right) h-ideal of S and both $\{0, a\}$ and $\{0, a, b, c\}$ are idempotent. Hence S is h-quasi-hemiregular.

The following lemma is obvious.

Lemma 6.4. *A hemiring S is left h-quasi-hemiregular is if and only if one of the following holds:*
(1) *There exist $c_i, d_i, c'_j, d'_j, z \in S$ such that*

$$x + \sum_{i=1}^{m} c_i x d_i x + z = \sum_{j=1}^{n} c'_j x d'_j x + z$$

for all $x \in S$;
(2) *$x \in \overline{SxSx}$ for all $x \in S$;*
(3) *$A \subseteq \overline{SASA}$ for all $A \in S$;*
(4) *$I \cap L = \overline{IL}$ for every h-ideal I and every left h-ideal L of S.*

Theorem 6.5. *A hemiring S is left (resp., right) h-quasi-hemiregular if and only if every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp., right) h-ideal is $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy idempotent.*

Proof. Let S be a left h-quasi-hemiregular hemiring, μ any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S. For $x \in S$. Since S is h-quasi-hemiregular, then by Lemma 6.4, there exist $c_i, c'_j, d_i, d'_j, z \in S$ such that

$$x + \sum_{i=1}^{m} c_i x d_i x + z = \sum_{j=1}^{n} c'_j x d'_j x + z.$$

Thus, we have

$$\begin{aligned}
(\mu \odot_h \mu)(x) \vee \gamma &= \bigvee_{x + \sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a'_j b'_j + z'} (\mu(a_i) \wedge \mu(a'_j) \wedge \mu(b_i) \wedge \mu(b'_j)) \vee \gamma \\
&\geq (\mu(c_i x) \wedge \mu(c'_j x) \wedge \mu(d_i x) \wedge \mu(d'_j x)) \vee \gamma \\
&= (\mu(c_i x) \vee \gamma) \wedge (\mu(c'_j x) \vee \gamma) \wedge (\mu(d_i x) \vee \gamma) \wedge (\mu(d'_j x) \vee \gamma) \\
&\geq (\mu(x) \wedge \delta) \wedge (\mu(x) \wedge \delta) \\
&= \mu(x) \wedge \delta,
\end{aligned}$$

which implies, $\mu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \mu$. Since μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S, we have $\mu \odot_h \mu \subseteq \vee q_{(\gamma,\delta)} \mu$. Whence, $\mu \sim \mu \odot_h \mu$.

Conversely, let L be any left h-ideal of S, then χ_L of L is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S. Thus,

$$\chi_L \sim \chi_L \odot_h \chi_L = \chi_{\overline{L^2}},$$

which implies, $L = \overline{L^2}$. Hence S is left h-quasi-hemiregular. □

The case for the $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideals can be proved similarly.

Theorem 6.6. *Let S be a hemiring. Then the following are equivalent:*

(1) *S is left h-quasi-hemiregular;*

(2) *$\mu \cap \nu \sim \mu \odot_h \nu$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal ν of S;*

(3) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal ν of S;*

(4) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal ν of S.*

Proof. (1) $\Longrightarrow$ (3): Let μ and ν be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal and any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal of S, respectively. For $x \in S$. By Lemma 6.4, we have $x \in \overline{SxSx} \subseteq \overline{S\overline{SxSx}Sx} \subseteq \overline{SxSxSx}$, and so there exist $c_i, c_j', d_i, d_j', e_i, e_j', z \in S$ such that

$$x + \sum_{i=1}^{m'} c_i x d_i x e_i x + z = \sum_{j=1}^{n'} c_j' x d_j' x e_j' x + z.$$

Thus, we have

$$\begin{aligned}
(\mu \odot_h \nu)(x) \vee \gamma &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a_j' b_j' + z'} (\mu(a_i) \wedge \mu(a_j') \wedge \nu(b_i) \wedge \nu(b_j')) \vee \gamma \\
&\geq (\mu(c_i x d_i) \wedge \mu(c_j' x d_j') \wedge \nu(x e_i x) \wedge \nu(x e_j' x)) \vee \gamma \\
&= (\mu(c_i x d_i) \vee \gamma) \wedge (\mu(c_j' x d_j') \vee \gamma) \wedge (\nu(x e_i x) \vee \gamma) \wedge (\nu(x d_j' x) \vee \gamma) \\
&\geq (\mu(x) \wedge \delta) \wedge (\nu(x) \wedge \delta) \\
&= \mu(x) \wedge \nu(x) \wedge \delta \\
&= (\mu \cap \nu)(x) \wedge \delta,
\end{aligned}$$

which implies, $\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu$. Thus (3) holds.

It is clear (3) $\Longrightarrow$ (4) $\Longrightarrow$(2).

(2) $\Longrightarrow$ (1): Let I and L be any h-ideal and any left h-ideal of S, respectively. Then, χ_I and χ_L are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal and $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S, respectively.

Thus,

$$\chi_{I\cap L} = \chi_I \cap \chi_L \sim \chi_I \odot_h \chi_L = \chi_{\overline{IL}},$$

which implies, $I \cap L = \overline{IL}$. It follows from Lemma 6.4 that S is h-quasi-hemiregular. □

Similarly, we can conclude the following result:

Theorem 6.7. *Let S be a hemiring. Then the following are equivalent:*

(1) *S is left h-quasi-hemiregular;*

(2) *$\mu \cap \nu \cap \omega \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu \odot_h \omega$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal ν and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal ω of S;*

(3) *$\mu \cap \nu \cap \omega \subseteq \vee q_{(\gamma,\delta)} \mu \odot_h \nu \odot_h \omega$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal ν and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal ω of S.*

Now, we can describe the characterization of h-quasi-hemiregular hemirings:

Theorem 6.8. *A hemiring S is h-quasi-hemiregular if and only if $\mu \sim (\chi_S \odot_h \mu)^2 \cap (\mu \odot_h \chi_S)^2$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S.*

Proof. Let S be an h-quasi-hemiregular hemiring, and μ any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S. We know $\chi_S \odot_h \mu$ and $\mu \odot_h \chi_S$ are an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal and an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy right h-ideal of S, respectively, and so both $\chi_S \odot_h \mu$ and $\mu \odot_h \chi_S$ are $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy idempotent by Theorem 6.5. Hence, we have

$$(\chi_S \odot_h \mu)^2 \cap (\mu \odot_h \chi_S)^2 \sim (\chi_S \odot_h \mu) \cap (\mu \odot_h \chi_S) \subseteq \vee q_{(\gamma,\delta)} \mu.$$

For $x \in S$. Since S is left h-quasi-hemiregular, then there exist $c_i, c_j', d_i, d_j', z \in S$ such that

$$x + \sum_{i=1}^{m'} c_i x d_i x + z = \sum_{j=1}^{n'} c_j' x d_j' x + z.$$

Thus, we have

$$\begin{aligned}(\chi_S \odot_h \mu)^2(x) \vee \gamma &= \bigvee_{x+\sum_{i=1}^{m} a_i b_i + z' = \sum_{j=1}^{n} a'_j b'_j + z'} ((\chi_S \odot_h \mu)(a_i) \wedge (\chi_S \odot_h \mu)(a'_j) \wedge (\chi_S \odot_h \mu)(b_i) \wedge (\chi_S \odot_h \mu)(b'_j)) \vee \gamma \\ &\geq ((\chi_S \odot_h \mu)(c_i x) \wedge (\chi_S \odot_h \mu)(c'_j x) \wedge (\chi_S \odot_h \mu)(d_i x) \wedge (\chi_S \odot_h \mu)(d'_j x)) \vee \gamma \\ &\geq (\mu \cap \nu)(x) \wedge \delta,\end{aligned}$$

which implies, $\mu \subseteq \vee q_{(\gamma,\delta)}(\chi_S \odot_h \mu)^2$. Similarly, we can prove $\mu \subseteq \vee q_{(\gamma,\delta)}(\mu \odot_h \chi_S)^2$, and so, $\mu \subseteq \vee q_{(\gamma,\delta)}(\chi_S \odot_h \mu)^2 \cap (\mu \odot_h \chi_S)^2$. Thus, $\mu \sim (\chi_S \odot_h \mu)^2 \cap (\mu \odot_h \chi_S)^2$. □

Conversely, assume that the given condition holds. Let μ be any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal of S, then by Proposition 4.7, μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal of S.

Thus, $\mu \sim (\chi_S \odot_h \mu)^2 \cap (\mu \odot_h \chi_S)^2 \subseteq \vee q_{(\gamma,\delta)}(\chi_S \odot_h \mu)^2 \subseteq \vee q_{(\gamma,\delta)}(\mu \odot_h \mu)^2 \subseteq \vee q_{(\gamma,\delta)}(\chi_S \odot_h \mu) \subseteq \vee q_{(\gamma,\delta)}\mu$.

Which implies, $\mu \sim \mu \odot_h \mu$. By Theorem 6.5, S is left h-quasi-hemiregular. Similarly, we can prove S is right h-quasi-hemiregular. Thus, S is h-quasi-hemiregular.

Similar to Lemma 6.4, we have the following:

Lemma 6.9. *A hemiring S is both left h-quasi-hemiregular and h-intra-hemiregular if and only if for any $x \in S$, there exist $c_i, d_i, c'_j, d'_j, z \in S$ such that $x + \sum_{i=1}^{m} c_i x^2 d_i x + z = \sum_{j=1}^{n} c'_j x^2 d'_j x + z$.*

Similar to Theorem 6.6, we have the following:

Theorem 6.10. *Let S be a hemiring. Then the following are equivalent:*

(1) *S is both left h-quasi-hemiregular and h-intra-hemiregular;*

(2) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)}\mu \odot_h \nu$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-bi-ideal ν of S;*

(3) *$\mu \cap \nu \subseteq \vee q_{(\gamma,\delta)}\mu \odot_h \nu$, for every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left h-ideal μ and every $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy h-quasi-ideal ν of S.*

7. Conclusions

In the study of a fuzzy algebraic system, we notice that the (fuzzy) ideals with special properties play an important role. By some kinds of fuzzy h-ideals, we investigate some characterizations of the h-intra-hemiregular and h-quasi-hemiregular hemirings.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of hemirings. In the future study of fuzzy hemirings, perhaps the following topics are worth to be considered:

(1) To establish the n-ary-hemirings;
(2) To describe soft n-ary-hemirings;
(3) To discuss fuzzy soft n-ary-hemirings and its applications;
(4) To consider these results to some possible applications in computer sciences and information systems in the future.

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